Derangements and asymptotics of Laplace transforms of polynomials.

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Abstract

We describe the behavior as $n \to \infty$ of the Laplace transforms of P^n, where P a fixed complex polynomial. As a consequence we obtain a new elementary proof of an result of Gillis-Ismail-Offer [2] in the combinatorial theory of derangements.

1 Statement of the main results

The generalized derangement problem in combinatorics can be formulated as follows. Suppose X is a finite set and $\sim$ is an equivalence relation on X. For each $x \in X$ we denote by $\hat{x}$ the equivalence class of x. $\hat{X}_\sim$ will denote the set of equivalence classes. The counting function of $\sim$ is the function

$$\nu = \nu_\sim : \hat{X} \to \mathbb{Z}, \;\; \nu(\hat{x}) = |\hat{x}|.$$

A $\sim$-*derangement* of x is a permutation $\varphi : X \to X$ such that

$$x \notin \hat{x}, \;\; \forall x \in X.$$

We denote by $\mathcal{N}(X, \sim)$ the number of $\sim$-derangements. The ratio

$$p(X, \sim) = \frac{\mathcal{N}(X, \sim)}{|X|!}$$

is the probability that a randomly chosen permutation of X is a derangement.

In [1] S. Even and J. Gillis have described a beautiful relationship between these numbers and the Laguerre polynomials

$$L_n(x) = e^x \frac{d^n}{dx^n}\Big(e^{-x} x^n\Big) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!}, \;\; n = 0, 1, \cdots .$$

For example

$$L_0(x) = 1, \;\; L_1(x) = 1 - x, \;\; L_2(x) = \frac{1}{2!}(x^2 - 4x + 2).$$

We set

$$L_{\sim} := \prod_{c\in\hat{X}} (-1)^{\nu(c)}\nu(c)! L_{\nu(c)}(x).$$

Observe that the leading coefficient of $L_{\sim}$ is 1. We have the following result.

Theorem 1.1 (Even-Gillis).

$$\mathcal{N}(X,\sim) = \int_0^\infty e^{-x} L_{\sim}(x)dx. \tag{1.1}$$

For a very elegant short proof we refer to [3].

Given $(X,\sim)$ as above and n a positive integer we define $(X_n,\sim_n)$ to be the disjoint union of n-copies of X

$$X_n = \bigcup_{k=1}^n X\times\{k\}$$

equipped with the equivalence relation

$$(x,j)\sim_n (y,k) \Longleftrightarrow j=k,\ \ x\sim y.$$

We deduce

$$p(X_n,\sim_n) = \frac{1}{(n|X|)!}\int_0^\infty e^{-x}\big(L_{\sim}(x)\big)^n dx \tag{1.2}$$

For example, consider the "marriage relation"

$$(C,\sim),\ \ C=\{\pm 1\},\ \ -1\sim 1.$$

In this case $\hat{C}$ consists of a single element and the counting function is the number $\nu=2$. Then $(C_n,\sim_n)$ can be interpreted as a group of n married couples. If we set

$$\delta_n := p(C_n,\sim_n)$$

then we can give the following amusing interpretation for δ_n.

Couples mixing problem. *At a party attended by n couples, the guests were asked to put their names in a hat and then to select at random one name from that pile. Then the probability that nobody will select his/her name or his/her spouse's name is equal to δ_n.*

Using (1.2) we deduce

$$\delta_n = \frac{1}{(2n)!}\int_0^\infty e^{-x}\big(x^2-4x+2\big)^n. \tag{1.3}$$

We can ask about the asymptotic behavior of the probabilities $p(X_n,\sim_n)$ as $n\to\infty$. In [2], Gillis-Ismail-Offer describe the first two terms of an asymptotic expansion in powers of n^{-1}. To formulate it let us introduce "momenta"

$$\nu_r = \sum_{c\in\hat{X}} \nu(c)^r.$$

Theorem 1.2 (Gillis-Ismail-Offer).

$$p(X_n, \sim_n) = \exp(-\frac{\nu_2}{\nu_1})\Big(1 - \frac{\nu_1(2\nu_3 - \nu_2) - \nu_2^2}{2\nu_1^3} n^{-1} + O(n^{-2}\Big) \ \ as \ n \to \infty. \tag{1.4}$$

For example we deduce from the above that

$$\delta_n = e^{-2}\Big(1 - \frac{1}{2}n^{-1} + O(n^{-2}\Big), \ \ n \to \infty \tag{1.5}$$

The proof in [2] of the asymptotic expansion (1.4) is based on the saddle point technique applied to the integrals in the RHS of (1.2) and special properties of the Laguerre polynomials.

In this paper we will investigate the large n asymptotics of Laplace transforms

$$\mathcal{F}_n(\mathcal{Q}, z) = \frac{z^{dn+1}}{(dn)!} \int_0^\infty e^{-zt} \mathcal{Q}(t)^n dt, \ \ \mathfrak{Re}\, z > 0, \tag{1.6}$$

where $\mathcal{Q}(t)$ is a degree d complex polynomial with leading coefficient 1. If we denote by $\mathcal{L}[f(t), z]$ the Laplace transform of $f(t)$

$$\mathcal{L}[f(t), z] = \int_0^\infty e^{-zt} f(t) dt \ \text{ then } \ \mathcal{F}_n(\mathcal{Q}, z) = \frac{\mathcal{L}[\mathcal{Q}(t)^n, z]}{\mathcal{L}[t^{dn}, z]}.$$

The estimate (1.4) will follow from our results by setting

$$z = 1, \ \ \mathcal{Q} = L_\sim.$$

To formulate the main result we first write $\mathcal{Q}$ as a product

$$\mathcal{Q}(t) = \prod_{i=1}^d (t + r_i).$$

We set

$$\vec{r} = (r_1, \cdots, r_d) \in \mathbb{C}^d, \ \ \mu_s = \mu_s(\vec{r}) = \frac{1}{d}\sum_{i=1}^d r_i^s$$

Theorem 1.3 (Existence theorem). *For every $\mathfrak{Re}\, z > 0$ we have an asymptotic expansion as $n \to \infty$*

$$\mathcal{F}_n(\mathcal{Q}, z) = (\sum_{k=0}^\infty A_k(z) n^{-k}\Big). \tag{1.7}$$

Above, the term $A_k(z)$ is a holomorphic function on $\mathbb{C}$ whose coefficients are universal elements in the ring of polynomials $\mathbb{C}(d)[\mu_1, \mu_2, \cdots, \mu_k]$, where $\mathbb{C}(d)$ denotes the field of rational functions in the variable $d = \deg \mathcal{Q}$.

We can say a bit more about the coefficients $A_k(z)$.

Theorem 1.4 (Structure Theorem). *For any k and any degree d we have*

$$A_k(z) = e^{\mu_1 z} B_k(z),$$

where $B_k \in \mathbb{C}(d)[\mu_1, \cdots, \mu_k][z]$ is a universal **polynomial** *in z with coefficients in $\mathbb{C}(d)[\mu_1, \cdots, \mu_k]$.*

Here is a brief description of the contents. We give the proof of the existence theorem in the next section, while in the third section we compute the terms A_k in some cases. For example we have

$$A_0(z) = e^{\mu_1 z}, \;\; A_1(z) = \frac{1}{2d} e^{\mu_1 z}(\mu_1^2 - \mu_2) z^2,$$

and we can refine (1.5) to

$$\delta_n = e^{-2}\Big(1 - \frac{1}{2} n^{-1} - \frac{23}{96} n^{-2} + O(n^{-2}\Big), \;\; n \to \infty. \tag{1.8}$$

These computations will lead to a proof of the structure theorem.

For the reader's convenience we include a list of symbols we will use throughout the paper. The symbol $[n]$ denotes the set $\{1, 2, \cdots, n\}$. A *d-dimensional (multi)index* will be a vector $\vec{\alpha} \in \mathbb{Z}^d_{\geq 0}$. For every vector $\vec{x} \in \mathbb{C}^d$ and any d-dimensional index $\vec{\alpha}$ we define

$$\vec{x}^{\vec{\alpha}} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \;\; |\vec{\alpha}| = \alpha_1 + \cdots + \alpha_d, \;\; S(\vec{x}) = x_1 + \cdots + x_d.$$

If $n = |\vec{\alpha}|$ then we define the multinomial coefficient

$$\binom{n}{\vec{\alpha}} := \frac{n!}{\prod_{i=1} \alpha_i!}.$$

These numbers appear in the *multinomial formula*

$$S(\vec{x})^n = \sum_{|\vec{\alpha}|=n} \binom{n}{\vec{\alpha}} \vec{x}^{\vec{\alpha}}.$$

2 Proof of the existence theorem

The key to our approach is the following elementary result.

Lemma 2.1. *If $P(x) = p_m t^m + \cdots p_1 t + p_0$ is a degree m with complex coefficients then for every $\mathfrak{Re}\, z > 1$ we have*

$$\frac{\mathcal{L}[P(t), z]}{\mathcal{L}[t^m, z]} = \frac{z^{m+1}}{m!} \int_0^\infty e^{-zt} P(t) dt = \sum_{a+b=m} \frac{p_a}{\binom{m}{a}} \frac{z^b}{b!}. \tag{2.1}$$

Proof

$$\frac{z^{m+1}}{m!} \int_0^\infty e^{-zt} P(t) dt = \frac{z^{m+1}}{m!} \sum_{a=0}^m p_a \int_0^\infty e^{-zt} t^a dt = \frac{z^{m+1}}{m!} \sum_{a=0}^m p_a \frac{a!}{z^{a+1}} = \sum_{a+b=m} \frac{p_a}{\binom{m}{a}} \frac{z^b}{b!}.$$

□

Denote by $\mathcal{Q}(n, a)$ the coefficient of t^a in $\mathcal{Q}(t)^n$. From (2.1) we deduce

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn} \frac{\mathcal{Q}(n, a)}{\binom{dn}{a}} \frac{z^b}{b!}. \tag{2.2}$$

Using the equality

$$\mathcal{Q}^n = \prod_{i=1}^d \underbrace{\left(\sum_{j+k=n}^n \binom{n}{i} t^j r_i^k \right)}_{(t+r_i)^n}$$

we deduce that if $a + b = dn$ then

$$\mathcal{Q}(n, a) = \sum_{|\vec{\alpha}|=b} \left(\prod_{i=1}^d \binom{n}{\alpha_j} \right) \vec{r}^{\alpha}. \tag{2.3}$$

For $|\vec{\alpha}| = b$ we set

$$B(n, \vec{\alpha}) := \prod_{i=1}^d \binom{n}{\alpha_j}, \;\; P_{n,b}(\vec{\alpha}) := \frac{B(n, \vec{\alpha})}{\binom{dn}{|\vec{\alpha}|}}, \;\; \rho_b(\vec{\alpha}) = \vec{r}^{\vec{\alpha}}.$$

so that

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn} \left(\sum_{|\vec{\alpha}|=b} P_{n,b}(\vec{\alpha}) \rho_b(\vec{\alpha}) \right) \cdot \frac{z^b}{b!} \tag{2.4}$$

Observe that we have

$$P_{n,b}(\vec{\alpha}) = \frac{\prod_{i=1}^d \underline{(}1 - \frac{1}{n}) \cdots (1 - \frac{\alpha_i - 1}{n})}{\prod_{k=1}^{b-1} (1 - \frac{k}{dn})} \cdot \underbrace{\frac{1}{d^b} \binom{b}{\vec{\alpha}}}_{:=P_b(\vec{\alpha})}. \tag{2.5}$$

The coefficients $P_b(\vec{\alpha})$ define the multinomial probability distribution P_b on the set of multiindices

$$\Lambda_b = \left\{ \vec{\alpha} \in (\mathbb{Z}_{\geq 0})^b; \;\; |\vec{\alpha}| = b \right\}$$

For every random variable ζ on Λ_b we denote by $E_b(\zeta)$ its expectation with respect to the probability distribution P_b. For each n we have a random variable $\zeta_{n,b}$ on Λ_b defined by

$$\zeta_{n,b}(\vec{\alpha}) = \frac{\prod_{i=1}^d \underline{(}1 - \frac{1}{n}) \cdots (1 - \frac{\alpha_i - 1}{n})}{\prod_{k=1}^{b-1} (1 - \frac{k}{dn})} \rho_b(\vec{\alpha}).$$

Form (2.4) and (2.5) we deduce

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn} E_b(\zeta_{n,b}) \frac{z^b}{b!}. \tag{2.6}$$

To find the asymptotic expansion for $\mathcal{F}_n$ we will find asymptotic expansions in powers of n^{-1} for the expectations $E_b(\zeta_{n,b})$ and them add them up using (2.6).

For every nonnegative integer α we define a polynomial

$$W_\alpha(x) = \begin{cases} 1 & \text{if} \quad \alpha = 0, 1 \\ \prod_{j=1}^{\alpha-1} (1 - jx) & \text{if} \quad \alpha > 1 \end{cases}$$

For a d-dimensional multi-index $\vec{\alpha}$ we set

$$W_{\vec{\alpha}}(x) = \prod_{i=1}^{d} W_{\alpha_i}(x).$$

We can now rewrite (2.5) as

$$P_{n,b}(\vec{\alpha}) = P_b(\vec{\alpha}) \frac{W_{\vec{\alpha}}(\frac{1}{n})}{W_b(\frac{1}{dn})}.$$

We set

$$R_b(\vec{\alpha}, x) = W_{\vec{\alpha}}(x), \quad K_b(\vec{\alpha}, x) = \frac{1}{W_b(\frac{x}{d})} R_b(\vec{\alpha}, x)\rho_b(\alpha).$$

We regard the correspondences

$$\vec{\alpha} \mapsto R_b(\vec{\alpha}, x), \; K_b(\vec{\alpha}, x)$$

as a random variables $R_b(x)$ and $K_b(x)$ on Λ_b valued in the field of rational functions. We deduce

$$\zeta_{n,b} = K_b(n^{-1}).$$

Observe

$$E_b(x) = E_b(K_b(x)) = \frac{1}{W_b(x)} E_b(R_b(x))$$

From the fundamental theorem of symmetric polynomials we deduce that the expectations $E_b(R_b(x))$ are *universal* polynomials

$$E_b(R_b(x)) \in \mathbb{C}[\mu_1, \cdots, \mu_b][x], \quad \deg_x E_b(R_b(x)) \leq b - d,$$

whose coefficients have degree b in the variables μ_i, $\deg \mu_i = i$. We deduce that $E_b(x)$ has a Taylor series expansion

$$E_b(x) = \sum_{m \geq 0} E_b(m) x^m$$

such that $E_b(m) \in \mathbb{C}(d)[\mu_1, \cdots, \mu_b]$. The rational function $x \to K_b(\vec{\alpha}, x)$ has a Taylor expansion at $x = 0$ convergent for $|x| < \frac{d}{b-1}$ so the above series converges for $|x| < \frac{d}{b-1}$. We would like to estimate the size of the coefficients $E_b(m)$. The tricky part is that the radius of convergence of $E_b(x)$ goes to zero as $b \to \infty$.

Lemma 2.2. *Set*

$$R = \max_{1 \leq i \leq d} |r_i|.$$

There exists a constant C which depends only on R and d such that for every $b \geq 0$ and every $1 \leq \lambda_b < \frac{b}{b-1}$ we have the inequality

$$|E_b(m)| = \Big(\frac{b}{\lambda_b d}\Big)^m C^b \frac{b^{b-1}}{(b-2)!(1 - \lambda_b \frac{b-1}{b})}. \tag{2.7}$$

Proof Note first that

$$|\rho_b(\vec{\alpha})| \leq R^b, \ \ \forall |\vec{\alpha}| = b.$$

For $b = 0, 1$ we deduce form the definition of the polynomials W_α that $E_b(x) = 1$. Fix m and $b > 1$. Using the Cauchy residue formula we deduce

$$E_b(m) = \frac{1}{2\pi\sqrt{-1}} \int_{|x|=\hbar} \frac{1}{x^{m+1}} E_b(x)dx, \ \ \hbar = \lambda_b \cdot \frac{d}{b}.$$

Hence

$$|E_b(m)| \leq \frac{1}{\hbar^m} \sup_{|x|=\hbar} |E_b(x)| \leq \frac{b^m R^b}{(\lambda_b d)^m \min_{|x|=\hbar} |W_b(x/d)|} \cdot \max_{|x|=\hbar} E_b(R_b(x)).$$

Next observe that

$$W_b(x/d) = (b-1)! \prod_{k=1}^{b-1} (\frac{1}{k} - x/d)), \ \ \hbar/d < 1/k, \forall k \leq b-1,$$

from which we conclude

$$\min_{|x|=\hbar} |W_b(x)| = W_b(\hbar) = \prod_{k=1}^{b-1} (1 - \frac{k\lambda_b}{b}) = \frac{1}{b^{b-1}} \prod_{k=1}^{b-1} (b - k\lambda_b) \geq \frac{(b-2)!(1 - \lambda_b \frac{b-1}{b})}{b^{b-1}}.$$

To estimate $E_b(R_b(x))$ from above observe that for every $1 \leq k \leq (b-1)$ and $|x| = \hbar$ we have

$$|1 - kx| \leq 1 + k|x| = 1 + \frac{k\lambda_b d}{b} < 1 + d$$

This shows that for every $|\vec{\alpha}| = b$ and $|x| = \hbar$ we have

$$|R_b(\vec{\alpha}, x)| < (1 + d)^b.$$

The lemma follows by assembling all the facts established above.

□

Define the formal power series

$$A_m(z) := \sum_{b \geq 0} E_b(m) \frac{z^b}{b!} \in \mathbb{C}\{z\}.$$

The estimate (2.7) shows that this series converges for all z.

For every formal power series $f = \sum_{k \geq 0} a_k T^k$ and every nonnegative integer ℓ we denote by $J_T^\ell(f)$ its ℓ-th jet

$$J_T^\ell(f) = \sum_{k=0}^{\ell} a_k T^k.$$

For $x = n^{-1}$ we have

$$\mathcal{F}_x(z) = \mathcal{F}_n(\mathcal{Q}, z) = \sum_{b \leq d/x} E_b(x) \frac{z^b}{b!} = \sum_{b \leq d/x} \Big(\sum_{m \geq 0} E_b(m) x^m \Big) \frac{z^b}{b!}$$

$$= \sum_{m\geq 0}\Big(\sum_{b\leq d/x} E_b(m)\frac{z^b}{b!}\Big)x^m = \sum_{m\geq 0} J_z^{d/x}(A_m(z))x^m.$$

Consider the formal power series in x with coefficients in the ring $\mathbb{C}\{z\}$ of convergent power series in z

$$\mathcal{F}_\infty(z) = \sum_{m\geq 0} A_m(z)x^m \in \mathbb{C}\{z\}[[x]].$$

We will prove that for every $\ell \geq 0$ and every $z \in \mathbb{C}$ we have

$$|\mathcal{F}_n(z) - J_x^\ell \mathcal{F}_\infty(z)| = O(n^{-\ell-1}), \ \text{ as } n \to \infty. \tag{2.8}$$

To prove this it is convenient to introduce the "rectangles"

$$D_{u,v} = \Big\{(b,m) \in (\mathbb{Z}_{\geq 0})^2;\ \ b \leq u,\ \ m \leq v.\Big\}$$

In this notation we have ($x = n^{-1}$)

$$\mathcal{F}_n(z) = \sum_{(b,m)\in D_{n,\infty}} E_b(m)x^m\frac{z^b}{b!},\ \ J_x^\ell\mathcal{F}_\infty(z) = \sum_{(b,m)\in D_{\infty,\ell}} E_b(m)x^m\frac{z^b}{b!}.$$

Then

$$\mathcal{F}_n(z) - J_x^\ell\mathcal{F}_\infty(z) = \underbrace{\sum_{b\leq dn}\Big(\sum_{m>\ell} E_b(m)x^m\Big)\frac{z^b}{b!}}_{S_1(n)} + \underbrace{\sum_{m\leq \ell}\Big(\sum_{b>dn} E_b(m)\frac{z^b}{b!}\Big)x^m}_{S_2(n)}.$$

We estimate each sum separately. Using (2.7) with a $\lambda_b > 1$ to be specified later we deduce

$$\Big|\sum_{m>\ell}|E_b(m)x^m| \leq \frac{C^b b^{b-1}}{(b-2)!(1-\lambda_b\frac{b-1}{b})}\sum_{m>\ell}\Big(\frac{bx}{\lambda_b d}\Big)^m.$$

The inequality $b \leq dn$ can be translated into $\frac{bx}{d} \leq 1$ so that the above series is convergent for $b \leq dn$ whenever $\lambda_b > 1$ so that

$$\Big|\sum_{m>\ell}|E_b(m)x^m| \leq \frac{C^b b^{b-1}}{(b-2)!(1-\lambda_b\frac{b-1}{b})}\Big(\frac{bx}{\lambda_b d}\Big)^{\ell+1}\frac{1}{1-\frac{bx}{\lambda_b d}}.$$

When $b \leq dn$ we have

$$1 - \frac{bx}{\lambda_b d} > 1 - \frac{1}{\lambda_b}.$$

If we choose

$$\lambda_b = \Big(\frac{b}{b-1}\Big)^{1/2}$$

we deduce

$$1 - \lambda_b\frac{b-1}{b}) = 1 - \Big(\frac{b-1}{b}\Big)^{1/2} \Longrightarrow \frac{1}{1-\lambda_b\frac{b-1}{b}} < b$$

and, since $\frac{bx}{\lambda_b d} \leq \frac{b}{d}x$,

$$\frac{1}{1-\frac{bx}{\lambda_b d}} < \frac{1}{1-\frac{1}{\lambda_b}} < 2b.$$

Using the inequalities

$$k! > \left(\frac{k}{e}\right)^k, \ \ \forall k > 0$$

we conclude that for $b \leq dn$ we have

$$\sum_{m>\ell} |E_b(m)x^m| \leq C_1^b b^{\ell+2} x^{\ell+1}.$$

Since the series $\sum_{b\geq 0} C_1^b b^{\ell+2} \frac{z^b}{b!}$ converges we conclude that

$$S_1(n) = O(x^{\ell+1}).$$

To estimate the second sum we choose $\lambda_b = 1$ in (2.7) and we deduce

$$E_b(m) \leq C_3^b.$$

Hence

$$\Big|\sum_{b>dn} E_b(m)\frac{z^b}{b!}| \leq \frac{(C_3|z|)^b b^2}{b!} < (2C_3|z|)^2 \sum_{b>dn} \frac{(|C_3|z|)^{b-2}}{(b-2)!}$$

Using Stirling's formula we deduce that for fixed z we have

$$\sum_{b>dn} \frac{(|C_3|z|)^{b-2}}{(b-2)!} < C_4(z) n^{-\ell-1}.$$

Hence

$$|S_2(n)| \leq C_4(z)(\ell+1)n^{-\ell-1}.$$

This completes the proof of (2.8) and of Theorem 1.3.

□

3 Additional structural results

§3.1 The case $d = 1$. Hence

$$\mathcal{Q}(t) = (t + \mu_1).$$

We have

$$\int_0^\infty e^{-zt}(t+\mu_1)^n dt = e^{\mu_1 z}\int_0^\infty e^{-zt} t^n dt = e^{\mu_1 z}\frac{n!}{z^{n+1}}$$

Hence in this case

$$\mathcal{F}_n(z) = e^{\mu_1 z}$$

and we deduce

$$A_0(z) = e^{\mu_1 z}, \ \ A_k(z) = 0, \ \ \forall k \geq 1.$$

§3.2 The case $d = 2$. This is a bit more complicated. We assume first that $\mu_1 = 0$ so that

$$\mathcal{Q}(t) = t^2 - \sigma^2$$

Then

$$\mathcal{Q}(n, a) = \begin{cases} (-1)^k \sigma^{2(n-k)} \binom{n}{k} & \text{if} \quad a = 2k \\ \\ 0 & \text{if} \quad a \text{ is odd} \end{cases},$$

and we deduce

$$\mathcal{F}_n(z) = \sum_{b=0}^{n} \frac{(-1)^b \binom{n}{n-b}}{\binom{2n}{2n-2b}} \frac{(\sigma z)^{2b}}{(2b)!} = \sum_{b=0}^{n} \frac{n!(2n-2b)!}{(n-b)!(2n)!} \frac{(-1)^b (\sigma z)^{2b}}{b!}$$

$$= \sum_{b=0}^{n} \frac{n(n-1)\cdots(n-b+1)}{2n(2n-1)\cdots(2n-2b+1)} \frac{(-1)^b (\sigma z)^{2b}}{b!}$$

$$= \sum_{b=0}^{n} \frac{1}{2^{2b}} n^{-b} \frac{(1-1/n)\cdots(1-(b-1)/n)}{(1-1/(2n)\cdots(1-(2b-1)/(2n))} \frac{(-1)^b (\sigma z)^{2b}}{b!}.$$

$$= 1 - \frac{1}{2} n^{-1} \frac{1}{1-1/(2n)} \frac{(\sigma z)^2}{2!} + \frac{1}{2^4} n^{-2} \frac{(1-1/n)}{(1-1/(2n))(1-2/(2n))(1-3/(2n))} \frac{(\sigma z)^4}{4!} + \cdots$$

To obtain $A_k(z)$ we need to collect the powers n^{-k}. The above description shows that the coefficients of the monomials z^{2b} contain only powers n^{-k}, $k \geq b$. We conclude that $A_k(z)$ is a polynomial and

$$\deg_z A_k(z) \leq 2k.$$

Let us compute the first few of these polynomials. We have

$$\mathcal{F}_n(z) = 1 - \frac{1}{2} n^{-1} \Big(1 + \frac{1}{2} n^{-1} + \cdots \Big) \frac{(\sigma z)^2}{2!} + \frac{1}{2^4} n^{-2} \Big(1 + \cdots \Big) \frac{(\sigma z)^4}{4!} + \cdots$$

We deduce

$$A_0(z) = 1, \;\; A_1(z) = -\frac{1}{4}(\sigma z)^2, \;\; A_2(z) = -\frac{1}{8}(\sigma z)^2 + \frac{1}{2^4 4!}(\sigma z)^4.$$

If $\mu_1 \neq 0$ so that

$$\mathcal{Q}(t) = (t + r_1)(t + r_2), \;\; r_1 + r_2 = 2\mu_1,$$

then we make the change in variables $t = s - \mu_1$ so that

$$\mathcal{Q}(t) = P(s) = s^2 - r^2, \;\; \sigma^2 = (r_1 - \mu_1)^2 = \frac{(r_1 - r_2)^2}{4}.$$

Now observe that

$$4\mu_1^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2 + (r_1 - r_2)^2 = 2(r_1^2 + r_2^2) = 4\mu_2$$

so that

$$\sigma^2 = \mu_2 - \mu_1^2.$$

Then

$$\mathcal{F}_n(\mathcal{Q}, z) = \frac{z^{2n+1}}{(2n)!}\int_0^\infty e^{-zt}\mathcal{Q}(t)^n = \frac{z^{2n+1}}{(2n)!}\int_0^\infty e^{-z(s-\mu_1)}P(s)^n ds = e^{\mu_1 z}\mathcal{F}_n(P, z).$$

We deduce

$$A_0(z) = e^{\mu_1 z},\ \ A_1(z) = -\frac{e^{\mu_1 z}}{4}(\sigma z)^2,\ \ A_2(z) = e^{\mu_1 z}\Big(-\frac{1}{8}(\sigma z)^2 + \frac{1}{2^4 4!}(\sigma z)^4\Big). \tag{3.1}$$

For the couples mixing problem we have

$$\mathcal{Q}(t) = t^2 - 4t + 2$$

so that

$$\mu_1 = -\frac{4}{2} = -2,\ \ \sigma^2 = \frac{1}{4}(r_1 - r_2)^2 = \frac{1}{4}\Big((r_1 + r_2)^2 - 4r_1 r_2\Big) = \frac{1}{4}(16 - 8) = 2.$$

and we deduce

$$\delta_n = \mathcal{F}_n(\mathcal{Q}, z = 1) = e^{-2}\Big(1 - \frac{1}{2}n^{-1} - \frac{23}{96}n^{-2} + O(n^{-3})\Big). \tag{3.2}$$

§3.3 The general case. Let us determine the coefficients $A_0(z)$ and $A_1(z)$ for general degree d. We use the definition

$$A_k(z) = \sum_{b\geq 0} E_b(k)\frac{z^b}{b!}.$$

For $|\vec{\alpha}| = b$

$$W_{\vec{\alpha}}(x) = W_{b,\alpha}(x) = \prod_{i=1}^d\Big(\prod_{j=1}^{\alpha_i - 1}(1 - jx)\Big)$$

$$= \prod_{i=1}^d\left(1 - \Big(\sum_{j=1}^{\alpha_i - 1} j\Big)x + \cdots\right) = 1 - \frac{1}{2}\Big(\sum_{i=1}^d \alpha_i(\alpha_i - 1)\Big)x + \cdots .$$

$$W_b(x/d) = \prod_{k=1}^{b-1}(1 + jx/d + \cdots) = 1 + \frac{b(b-1)}{2d}x + \cdots .$$

Next compute the expectation of $R_b(x)$

$$E_b(R_b(x)) = E_b(\rho_b) - \frac{1}{2}E_b\Big(\sum_{i=1}^d \alpha_i(\alpha_i - 1)\vec{r}^{\vec{\alpha}}\Big)x + \cdots .$$

The multinomial formula implies

$$E_b(\rho_b) = \mu_1^b.$$

Next

$$E_b\Big(\sum_{i=1}^d \alpha_i(\alpha_i - 1)\vec{r}^{\vec{\alpha}}\Big) = \frac{1}{d^b}\sum_{|\vec{\alpha}|=b}\binom{b}{\vec{\alpha}}\Big(\sum_{i=1}^d \alpha_i(\alpha_i - 1)\Big)\vec{r}^{\vec{\alpha}}$$

Now consider the partial differential operator

$$\mathcal{P} = \sum_{i=1}^{d} r_i^2 \frac{\partial^2}{\partial r_i^2}.$$

Observe that the monomials $\vec{r}^{\vec{\alpha}}$ are eigenvectors of $\mathcal{P}$

$$\mathcal{P}\vec{r}^{\vec{\alpha}} = (\sum_{i=1}^{d} \alpha_i(\alpha_i - 1))\vec{r}^{\vec{\alpha}}.$$

We deduce

$$E_b\Big(\sum_{i=1}^{d} \alpha_i(\alpha_i - 1)\vec{r}^{\vec{\alpha}}\Big) = \frac{1}{2d^b}\mathcal{P}S(\vec{r})^b = \frac{1}{2}\mathcal{P}\mu_1^b.$$

Hence

$$E_b(R_b(x) = \mu_1^b - \frac{1}{2}(\mathcal{P}\mu_1^b)x + \cdots$$

and we deduce

$$\begin{aligned} E_b(x) &= (\mu_1^b - \frac{1}{2}(\mathcal{P}\mu_1^b)x + \cdots)(1 + \frac{b(b-1)}{2d}x + \cdots) \\ &= \mu_1^b + \frac{1}{2}\Big(\frac{b(b-1)}{d}\mu_1^b - \mathcal{P}\mu_1^b\Big)x + \cdots \end{aligned}$$

We deduce $A_0(z) = e^{\mu_1 z}$

$$A_1(z) = \frac{\mu_1^2}{2d}\sum_{b=2}^{\infty}\frac{z^b}{(b-2)!} - \frac{1}{2}\mathcal{P}e^{\mu_1 z} = \frac{\mu_1^2 z^2}{2d}e^{\mu_1 z} - \frac{1}{2}\mathcal{P}e^{\mu_1 z}.$$

We can simplify the answer some more.

$$\mathcal{P}\mu_1^b = \frac{1}{d^b}\mathcal{P}S(x)^b = \frac{b(b-1)}{d^b}\Big(\sum_{i=1}^{d} r_i^2\Big)S(x)^{b-2} = \frac{b(b-1)}{d}\mu_2\mu_1^{b-2}.$$

We conclude that

$$\mathcal{P}e^{\mu_1 z} = \frac{\mu_2 z^2}{d}\sum_{b\geq 2}\frac{(\mu_1 z)^{b-2}}{(b-2)!} = \frac{\mu_2 z^2}{d}e^{\mu_1 z}.$$

Hence

$$A_0(z) = e^{\mu_1 z}, \;\; A_1(z) = \frac{e^{\mu_1 z}}{2d}(\mu_1^2 - \mu_2)z^2 \tag{3.3}$$

For $d = 2$ we recover part of the formulæ (3.1).

§3.4 The proof of Theorem 1.4 Clearly we can assume $d > 1$. We imitate the strategy used in the case $d = 2$. Thus, after the change in variables $t \to t - \mu_1$ we can assume that $\mu_1 = 0$ so that $\mathcal{Q}(t)$ has the special form

$$\mathcal{Q}(t) = t^d + a_{d-2}t^{d-2} + \cdots + a_0.$$

Set

$$T(n,b) := \frac{\mathcal{Q}(n, dn-b)}{\binom{dn}{dn-b}}$$

This is a power series in $x = n^{-1}$,

$$T(n,b) = T_b(x)\,|_{x=n^{-1}}, \;\; T_b(x) = \sum_{k\geq 0} T_b(k)x^k.$$

We have

$$A_k(z) = \sum_{b\geq 0} T_b(k)\frac{z^b}{b!},$$

and we need to prove that A_k is a polynomial for every k. We denote by $\ell(b)$ the order of of the first non-zero coefficient of $T_b(x)$,

$$\ell(b) = \min\{k \geq 0;\;\; T_b(k) \neq 0\}.$$

To prove the desired conclusion it suffices to show that

$$\lim_{b\to\infty} \ell(b) = \infty. \tag{3.4}$$

For every multiindex $\vec{\beta} = (\beta_d, \beta_{d-2}, \cdots, \beta_1, \beta_0)$ we set

$$L(\vec{\beta}) = d\beta_d + (d-2)\beta_{d-2} + \cdots + \beta_1.$$

We set $\vec{a} = (1, a_{d-2}, \cdots, a_1, a_0) \in \mathbb{C}^d$. We have

$$T(n,b) = \frac{1}{\binom{dn}{dn-b}} \cdot \sum_{|\vec{\beta}|=n, L(\vec{\beta})=dn-b} \binom{n}{\vec{\beta}}\vec{a}^{\vec{\beta}}. \tag{3.5}$$

Now observe that we have

$$d|\vec{\beta} - L(\vec{\beta}) = 2\beta_{d-2} + 3\beta_{d-3} + \cdots (d-1)\beta_1 + d\beta_0 = d|\vec{\beta}| - L(\vec{\beta}) = b.$$

In particular we deduce

$$\beta_j \leq \frac{b}{d-j} \leq \frac{b}{2}, \;\; \forall 0 \leq j \leq d-2 \tag{3.6}$$

and

$$\begin{aligned} 2\beta_d + b = 2\beta_d &= 2\beta_{d-2} + 3\beta_{d-3} + \cdots (d-1)\beta_1 + d\beta_0 \\ &\geq 2\beta_d + 2\beta_{d-2} + \cdots + 2\beta_1 + 2\beta_0 = 2n \end{aligned}$$

so that

$$n - \beta_d \leq \frac{b}{2}. \tag{3.7}$$

These simple observations have several important consequences.

First, observe that they imply that there exists an integer $N(b)$ *which depends only b and d*, such that for any $n > 0$ the number of multi-indices $\vec{\beta}$ satisfying $|\vec{\beta}| = n$ and $L(\vec{\beta}) = dn - b$ is $\leq N(b)$. Thus the sum (3.5) has fewer than $N(b)$ terms.

Next, if we set $|a| := \max_{0 \leq j \leq d-2} |a_j|$ then, we deduce

$$|\vec{a}^{\vec{\beta}}| \leq |a|^{\beta_0 + \cdots + \beta_{d-2}} \leq |a|^{\frac{b(d-1)}{2}} = C_5(b).$$

Finally, using the identity

$$\binom{n}{\vec{\beta}} = \binom{n}{\beta_d} \cdot \binom{n-\beta_d}{\beta_{d-2}} \binom{n-\beta_d-\beta_{d-2}}{\beta_{d-3}} \cdots$$

the inequalities (3.7) and $\binom{m}{k} \leq 2^m$, $\forall m \geq k$ we deduce

$$\binom{n}{\vec{\beta}} \leq \binom{n}{\beta_d} \cdot 2^{\frac{b(d-1)}{2}} \leq 2^{\frac{b(d-1)}{2}} \binom{n}{\lfloor b/2 \rfloor + 1} \leq C_6(b) n^{\lfloor b/2 \rfloor + 1}, \ \ \forall n \gg b.$$

Hence

$$\sum_{|\vec{\beta}|=n, L(\vec{\beta})=dn-b} \left| \binom{n}{\vec{\beta}} \vec{a}^{\vec{\beta}} \right| \leq N(b) C_5(b) C_6(b) n^{\lfloor b/2 \rfloor + 1} = C_7(b) n^{\lfloor b/2 \rfloor + 1}.$$

On the other hand

$$\frac{1}{\binom{dn}{dn-b}} \leq C_8(b) n^{-b}$$

so that

$$|T(n,b)| = |T_b(n^{-1})| \leq C_9(b) n^{\lfloor b/2 \rfloor + 1 - b} \leq C_9(b) n^{1-b/2}.$$

This shows

$$T_b(k) = 0, \ \ \forall k \leq b/2 - 1$$

so that

$$\ell(b) \geq b/2 - 1 \to \infty \ \text{ as } b \to \infty.$$

□

Remark 3.1. We can say a bit more about the structure of the polynomials

$$B_k(\mu_1, \cdots, \mu_d, z) \in R_d = \mathbb{C}[\mu_1, \cdots, \mu_d, z], \ \ k > 0.$$

If we regard B as a polynomial in $r_1, \cdots, r_d$ we see that it vanishes precisely when $r_1 = \cdots = r_d$. Note that

$$r_1 = \cdots = r_d = r \Longleftrightarrow \mathcal{Q}(t) = (t + r)^d.$$

On the other hand

$$\sum_k t^k \mu_k = \frac{1}{d} \sum_{i=1}^d \sum_{k \geq 0} (r_i t)^k = \frac{1}{d} \sum_{i=1}^d \frac{1}{1 - r_i t} \overset{(s:=1/t)}{=} \frac{s}{d} \sum_{i=1}^d \frac{1}{s + \mu_i} = \frac{s}{d} \frac{\mathcal{Q}'(s)}{\mathcal{Q}(s)}.$$

If $\mathcal{Q}(s) = (s+r)^d$ we deduce

$$\frac{s}{d}\frac{\mathcal{Q}'(s)}{\mathcal{Q}(s)} = \frac{s}{s+r} = \frac{1}{1-rt} = \sum_{k\geq 0}(rt)^k.$$

This implies that

$$r_1 = \cdots = r_d \Longleftrightarrow \mu_i^j = \mu_j^i,\ \ \forall 1 \leq i,j \leq k \Longleftrightarrow \mu_j = \mu_1^j,\ \ \forall 1 \leq j \leq d.$$

The ideal I in R_d generated by the binomials $\mu_1^j - \mu_j$ is prime since $R_d/I \cong \mathbb{C}[\mu_1, z]$. Using the Hilbert *Nullstelensatz* we deduce that B_k must belong to this ideal so that we can write

$$B_k(\mu_1, \cdots, \mu_d, z) = A_{2k}(\mu, z)(\mu_1^2 - \mu_2) + \cdots + A_{dk}(\mu, z)(\mu_1^d - \mu_d).$$

$\square$

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